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A METHOD FOR ANALYZING NONLINEAR CLOSED-LOOP CONTROL SYSTEMS WITH STATIONARY STOCHASTIC INPUTS*

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SUMMARY

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This study presents a method for investigating the response of a closed-loop nonlinear control system subject to a stationary stochastic input with additive zero mean white Gaussian noise. An analytic technique for obtaining an approximate expression for the autocorrelation functions at the input and output of the nonlinearity, as well as other parts of the system's loop, is proposed. This analysis leads to an expression for the root-mean-square error between the input and output of the system.

Author

*A thesis submitted in July 1963 to the Department of Electrical Engineering and the Committee on the Graduate Division of Stanford University in partial fulfillment of the requirements for the degree of Engineer.

INTRODUCTION

At present, the principal method used to analyze a closed-loop nonlinear system subject to a stationary stochastic input is a statistical linearization referred to as an equivalent gain technique. In this method, the nonlinearity is replaced by an amplifier gain chosen so as to minimize the mean-square error between this linear representation and the actual response of the component. Although this method is easy to apply, it ignores the distortion that the nonlinearity produces in the autocorrelation functions of the system.

It has been shown that when a nonlinear device is subject to a Gaussian distributed signal, its output autocorrelation function can be expressed as a power series in terms of the input autocorrelation using Mehler's expansion.[1] The first term of this series corresponds to a linear gain and the others account for the distortion produced by the device. By use of a rectangular-pulse approximation for the autocorrelation functions of a closed-loop system, this paper shows that the Mehler expansion can be used to evaluate these functions and to obtain an accurate estimate of the root-mean-square error between the system's input and output. Matrix methods are used in this analysis. A digital computer can be used for matrix inversion to obtain numerical and analytical results expediently.

The method is first applied to a system which is of particular interest to the Ames Research Center of the National Aeronautics and Space Administration. The method is then extended to apply to the analysis of a broad class of closed-loop nonlinear systems.

STATEMENT OF INTERCEPTION PROBLEM

The particular system analyzed is an analog of what is commonly referred to as an interception problem. The system's input is the position of an airborne target contaminated by additive noise, and its output is the position of the pursuing aircraft. The forward loop of the system represents the dynamics of the pursuer, and includes an element to simulate the physical limitations imposed on the thrust capabilities of the plant. The remainder of this section will be devoted to specifying this system and its inputs. The fact that a real physical process is being represented will be implied in the definitions which are made.

The acceleration of the target has two values $+A_o$ and $-A_o$ and makes independent random transversals from one state to the other. The probability that k transversals will occur in time τ is given by a Poisson distribution:

$$P(k;\tau) = \frac{(K\tau)^k}{k!}\exp(-K\tau) \tag{1}$$

where K is the average number of transversals which occur per unit time, and $\tau \geq 0$.

Let the acceleration of the target at time t be represented by $x(t)$. The ensemble average of $x(t)x(t+\tau)$ is the expected value $\langle x(t)x(t+\tau)\rangle$, and is referred to as the autocorrelation function of $x(t)$, $R_x(\tau)$. If ergodicity is assumed in this analysis, this average is also the time average. From the statistics of $x(t)$, and for $\tau \geq 0$,

$$R_x(\tau) = (A_o)(A_o) \sum_{k \text{ even}} P(k;\tau) + (-A_o)(A_o) \sum_{k \text{ odd}} P(k;\tau)$$

$$= A_o^2 \exp(-K\tau) \sum_{k=0}^{\infty} \frac{(-K\tau)^k}{k!} = A_o^2 \exp(-2K\tau)$$

For $\tau < 0$, the autocorrelation function is obtained from this expression by replacing τ by $-\tau$. Hence, for all values of τ,

$$R_x(\tau) = A_o^2 \exp(-2K|\tau|) \tag{2}$$

As will be true for all autocorrelation functions,

$$R_x(\tau) = R_x(-\tau) \tag{3}$$

The noise is a Gaussian distributed random process with zero mean and standard deviation of A_1. It is referred to as being white, as its frequency distribution is uniform over the infinite domain. Because of its white characteristic, the noise amplitudes at two different instants of time are uncorrelated and, since it has zero mean,

$$R_n(\tau) = \langle n(t)n(t + \tau)\rangle = A_1^2\delta(\tau) \tag{4}$$

where $\delta(\tau)$ is the Kronecker delta.

Because the noise is white with zero mean, the cross-correlation function between noise and acceleration input, $R_{nx}(\tau)$, vanishes; that is,

$$R_{nx}(\tau) = \langle n(t)x(t + \tau)\rangle = \langle n(t)\rangle \langle x(t + \tau)\rangle = 0$$

For real stationary processes $\varphi(t)$ and $\theta(t)$ in general,

$$R_{\varphi\theta}(\tau) = \langle\varphi(t)\theta(t + \tau)\rangle = \langle\theta(t - \tau)\varphi(t)\rangle = R_{\theta\varphi}(-\tau) \qquad (5)$$

The two-sided Laplace transform is used in the analysis that follows. The transform of an autocorrelation function, $R_x(\tau)$, is a spectral density, $\mathbf{S}_x(s)$, and that of cross-correlation function, $R_{nx}(\tau)$, is cross-spectral density, $\mathbf{S}_{nx}(s)$. The two-sided Laplace transform pair that is used is defined by:

$$\mathbf{S}(s) = \mathbf{L}R(\tau) = \int_{-\infty}^{\infty} R(\tau)\, \exp(-s\tau)d\tau \qquad (6)$$

$$R(\tau) = \mathbf{L}^{-1}\mathbf{S}(s) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \mathbf{S}(s)\, \exp(s\tau)ds \qquad \text{where } \sqrt{-1} = j \qquad (7)$$

By the use of equations (2) and (6)

$$\begin{aligned}
\mathbf{S}_x(s) &= \int_{-\infty}^{\infty} R_x(\tau)\, \exp(-s\tau)d\tau \\
&= A_o^2 \left\{ \int_0^{\infty} \exp[-(s + 2K)\tau]d\tau + \int_{-\infty}^{0} \exp[-(s - 2K)\tau]d\tau \right\} \\
&= A_o^2 \left(\frac{1}{s + 2K} + \frac{1}{-s + 2K} \right) \\
&= \frac{2(2K)}{[(2K)^2 - s^2]}
\end{aligned}$$

or

$$\mathbf{S}_x(s) = \frac{r(s)r(-s)}{\gamma(s)\gamma(-s)} = \frac{K_o}{\gamma(s)\gamma(-s)} \qquad (8)$$

where

$$r(s) = \sqrt{K_o} = \sqrt{(4K)A_o^2}$$

$$\gamma(s) = s + 2K$$

and

$$\mathbf{S}_n(s) = \int_{-\infty}^{\infty} A_1^2\delta(\tau)\ \exp(-s\tau)d\tau = A_1^2 = K_1\ , \qquad \text{for all} \quad s \tag{9}$$

The symmetry relationships applicable to the spectral and cross-spectral densities are analogous to those pertaining to their inverse transformed quantities. For real stationary processes $\varphi(t)$ and $\theta(t)$, these relationships are:

$$\mathbf{S}_\varphi(s) = \mathbf{S}_\varphi(-s) \tag{10a}$$

$$\mathbf{S}_{\varphi\theta}(s) = \mathbf{S}_{\theta\varphi}(-s) \tag{10b}$$

The analog of the uncompensated plant dynamics is a static plant gain, G_o, which is followed by a "hard limiter," which is followed by a double integrator. These components are in the forward loop of the system. The static response of the limiter is:

$$y(t) = f[u(t)] = \begin{cases} V_o & \text{for} \quad u(t) \geq V_o \\ u(t) & \text{for} \quad |u(t)| < V_o \\ -V_o & \text{for} \quad u(t) \leq -V_o \end{cases} \tag{11}$$

A Wiener filter, designed to minimize the mean-square error between the system's input and output, precedes the uncompensated plant in the forward loop. The equations from which this filter was calculated are those of an equivalent linear system. They were obtained by replacing the nonlinearity with an amplifier gain selected so as to minimize the mean-square error between its response and that of the actual device. The transfer function of the Wiener filter is:

$$H'(s) = \frac{\eta'(s)}{\nu(s)} \tag{12}$$

A unity feedback path is placed from the output of the system to its input node, and the output is subtracted from the input to form the error signal.

The features discussed are indicated in the block diagram of the system. In this block diagram and in the presentation which follows, small English letters indicate the analog signals. The same letter, but with different argument notation, indicates the signal in the time domain and its Laplace transform. The system can be diagramed as follows:

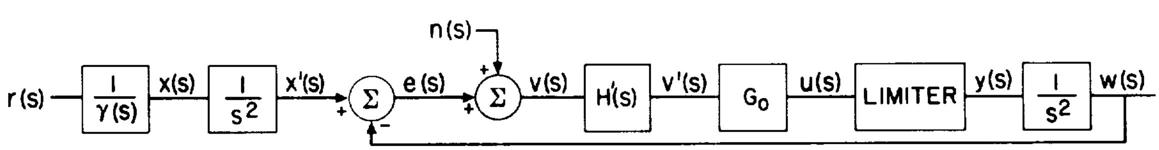

Figure 1.- Block diagram of a system corresponding to an interception problem.

The values of the parameters indicated above complete the statement of the problem. They are:

$$A_o = 12.43 \tag{13a}$$

$$2K = 0.4311 \tag{13b}$$

$$K_o = 4KA_o^2 = 133.2 \tag{13c}$$

$$K_1 = A_1^2 = 15.00 \tag{13d}$$

$$G_o = 370.0 \tag{13e}$$

$$V_o = 92.34 \tag{13f}$$

$$\gamma(s) = s + 2K = s + 0.4311 \tag{13g}$$

$$\eta(s) = 7.875 \times 10^{-2}(s^2 + 1.136s + 0.4773) \tag{13h}$$

$$\nu(s) = s^3 + 4.915s^2 + 12.01s + 4.448 \tag{13i}$$

ANALYSIS OF INTERCEPTION PROBLEM

If

$$\eta(s) = G_o\eta'(s) \tag{14a}$$

$$H(s) = G_oH'(s) \tag{14b}$$

and the block diagram of figure 1 is rearranged, the following equivalent system representation is obtained:

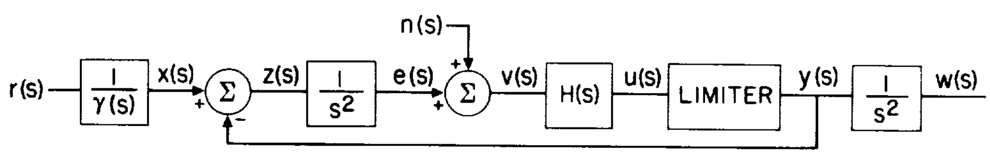

Figure 2.- Block diagram of an equivalent system corresponding to an interception problem.

The transforms of the analog signals of this system are related as follows:

$$z(s) = \frac{r(s)}{\gamma(s)} - y(s)$$

$$v(s) = e(s) + n(s) = \frac{z(s)}{s^2} + n(s)$$

Combining these equations, using

$$u(s) = H(s)v(s)$$

yields

$$H(s)\left[\frac{1}{s^2\gamma(s)} r(s) + n(s)\right] = u(s) + \frac{1}{s^2} H(s)y(s) \tag{15}$$

Since $x(t)$ and $n(t)$ are uncorrelated, the following relationship is valid:

$$H(s)H(-s)\left[\frac{1}{\gamma(s)\gamma(-s)s^4} \mathbf{L}\langle r(t)r(t+\tau)\rangle + \mathbf{L}\langle n(t)n(t+\tau)\rangle\right]$$

$$= \mathbf{L}\langle u(t)u(t+\tau)\rangle + \frac{H(s)}{s^2}\mathbf{L}\langle u(t)y(t+\tau)\rangle + \frac{H(-s)}{s^2}\mathbf{L}\langle y(t)u(t+\tau)\rangle$$

$$+ \frac{H(s)H(-s)}{s^4}\mathbf{L}\langle y(t)y(t+\tau)\rangle$$

The notation developed previously and equations (8) and (9) may be used to write this relationship as:

$$H(s)H(-s)\left[\frac{K_0}{s^4\gamma(s)\gamma(-s)} + K_1\right] = \mathbf{S}_u(s) + \frac{H(s)}{s^2}\mathbf{S}_{uy}(s) + \frac{H(-s)}{s^2}\mathbf{S}_{yu}(s)$$

$$+ \frac{H(s)H(-s)}{s^4}\mathbf{S}_y(s) \tag{16}$$

In the appendix it is shown that when the input to a nonlinear device is Gaussian distributed, the autocorrelation function at the output can be expressed as a power series in terms of the normalized input autocorrelation function using Mehler's expansion. In the situation being considered the nonlinearity is a limiter. This device truncates the probability distribution at its input and presents the distorted signal to the input node of the system by way of the feedback path. It might seem that, because of this distortion and the physically non-Gaussian nature of the input $x(t)$, the conditions under which the Mehler expansion is valid are not satisfied here.

Under a hypothesis based on the Central Limit theorem, the probability distribution of the limiter's input can be regarded as being quite close to Gaussian. This is due to the low pass characteristics of the linear system consisting of the double integrator and Wiener filter which precedes the limiter in the forward loop of the over-all equivalent system. The referred to hypothesis states that: "The asymptotic approach to normality of the output of a linear system whose input is not normal is most pronounced if the system function has low pass characteristics."[2]

If it is assumed, then, that the input to the limiter is Gaussian and the results of equations (A11) and (A12) are used, the following relationships are valid:

$$R_{yu}(\tau) = R_{uy}(\tau) = a_1\sigma_u^2\rho_u(\tau) = a_1R_u(\tau) \tag{17}$$

$$R_y(\tau) = \sigma_u^2 \sum_{n=0}^{\infty} a_n^2\rho_u^n(\tau) \tag{18}$$

where

$$a_n = \frac{1}{\sqrt{2\pi}\,\sigma_u} \int_{-\infty}^{\infty} f(\alpha\sigma_u)\, \chi_n(\alpha)\, \exp\left(\frac{-\alpha^2}{2}\right) d\alpha \tag{19}$$

$$\chi_n(\alpha) = \frac{(-1)^n}{\sqrt{n!}} \exp\left(\frac{\alpha^2}{2}\right) \frac{d^n}{d\alpha^n}\left[\exp\left(\frac{-\alpha^2}{2}\right)\right] \text{ is a Hermite polynomial} \tag{20}$$

$$\sigma_u^2 = R_u(0) \tag{21}$$

$$\rho_u(\tau) = \frac{R_u(\tau)}{\sigma_u^2} \tag{22}$$

Substituting the result of equation (17) and

$$H(s) = \frac{\eta(s)}{\nu(s)}$$

obtained from equations (12), (14a), and (14b), above, yields:

$$\frac{\eta(s)\eta(-s)}{\nu(s)\nu(-s)}\left[\frac{K_o}{s^4\gamma(s)\gamma(-s)} + K_1\right] = \mathbf{S}_u(s)\left\{1 + \frac{a_1}{s^2}\left[\frac{\eta(s)}{\nu(s)} + \frac{\eta(-s)}{\nu(-s)}\right]\right\}$$

$$+ \frac{\eta(s)\eta(-s)}{s^4\nu(s)\nu(-s)}\,\mathbf{S}_y(s)$$

or

$$\begin{aligned}
\frac{\eta(s)\eta(-s)}{\gamma(s)\gamma(-s)}[K_o + K_1 s^4\gamma(s)\gamma(-s)] &= \mathbf{S}_u(s)\Big\{s^4\nu(s)\nu(-s) + a_1 s^2[\eta(s)\nu(-s) \\
&\quad + \eta(-s)\nu(s)]\Big\} + \eta(s)\eta(-s)\mathbf{L}R_y(\tau) \\
&= \mathbf{S}_u(s)\Big\{s^4\nu(s)\nu(-s) + s^2\alpha_1[\eta(s)\nu(-s) \\
&\quad + \eta(-s)\nu(s)] + \alpha_1^2\eta(s)\eta(-s)\Big\} \\
&\quad + \mathbf{S}_u(s)\Big\{s^2(a_1 - \alpha_1)[\eta(s)\nu(-s) \\
&\quad + \eta(-s)\nu(s)] - \alpha_1^2\eta(s)\eta(-s)\Big\} \\
&\quad + \eta(s)\eta(-s)\mathbf{L}R_y(\tau)
\end{aligned}$$

where α_1 is a constant which is chosen during the computation process.

Defining

$$\xi(s) = s^2\nu(s) + \alpha_1\eta(s) = s^5 + 4.915s^4 + 12.01s^3$$
$$+(29.14\alpha_1 + 4.448)s^2 + 33.01\alpha_1 s + 13.91\alpha_1 \tag{23}$$

$$\mu(s) = [\eta(s)\nu(-s) + \eta(-s)\nu(s)]^+ = 14.84(s^2 + 1.820s + 0.7495) \tag{24}$$

where

$$\mu(s)\mu(-s) = \{\eta(s)\nu(-s) + \eta(-s)\nu(s)\}^+\{\eta(s)\nu(-s) + \eta(-s)\nu(s)\}^-$$
$$= [\eta(s)\nu(-s) + \eta(-s)\nu(s)]$$

yields

$$\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)\gamma(s)\gamma(-s)}\,[K_0 + K_1 s^4\gamma(s)\gamma(-s)]$$
$$= \mathbf{S}_u(s)\left[1 + \frac{(a_1 - \alpha_1)s^2\mu(s)\mu(-s) - \alpha_1^2\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right] + \frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\,\mathbf{L}R_y(\tau) \tag{25}$$

It can be shown that $\eta(s)/\xi(s)$ and $\mu(s)/\xi(s)$ each have all their poles and zeros in the right-half side of the s plane and are minimum phase functions.

It would be desirable to obtain a closed-form expression for:

$$R_y(\tau) = \sigma_u^2 \sum_{n=0}^{\infty} a_n^2 \rho_u^{\,n}(\tau)$$

Integrating equation (19) by parts, using equations (11) and (20), yields

$$a_n = \frac{(-1)^n}{\sigma_u \sqrt{2\pi(n!)}} \left\{ f(\alpha\sigma_u) \frac{d^{n-1}}{d\alpha^{n-1}} \exp\left(\frac{-\alpha^2}{2}\right) \right]_{\alpha=-\infty}^{\alpha=\infty}$$

$$- \int_{-\infty}^{\infty} \frac{d}{d\alpha} [f(\alpha\sigma_u)] \frac{d^{n-1}}{d\alpha^{n-1}} \left[\exp\left(\frac{-\alpha^2}{2}\right) \right] d\alpha \right\}$$

$$= \frac{(-1)^{n+1}}{\sqrt{2\pi(n!)}} \int_{-V_o/\sigma_u}^{V_o/\sigma_u} \frac{d}{d\alpha} [f(\alpha\sigma_u)] \frac{d^{n-1}}{d\alpha^{n-1}} \left[\exp\left(\frac{-\alpha^2}{2}\right) \right] d\alpha \ , \qquad n \geq 1$$

Since

$$\frac{d}{d\alpha} f(\alpha\sigma_u) = \begin{cases} \sigma_u \ , & |\alpha\sigma_u| < V_o \\ 0 \ , & |\alpha\sigma_u| \geq V_o \end{cases}$$

$$\frac{d^m}{d\alpha^m} \exp\left(\frac{-\alpha^2}{2}\right)\Bigg|_{\alpha=-\infty} = \frac{d^m}{d\alpha^m} \exp\left(\frac{-\alpha^2}{2}\right)\Bigg|_{\alpha=\infty} = 0$$

then

$$a_n = \frac{(-1)^{n+1}}{\sqrt{2\pi(n!)}} \left[\frac{d^{n-2}}{d\alpha^{n-2}} \exp\left(\frac{-\alpha^2}{2}\right) \right]_{\alpha=-\alpha_o}^{\alpha=\alpha_o} , \qquad n \geq 2 \tag{26}$$

where

$$\alpha_o = \frac{V_o}{\sigma_u} = \frac{92.34}{\sigma_u} \tag{27}$$

For $n \leq 2$ a_n is evaluated as:

$$a_o = \frac{1}{\sigma_u \sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha\sigma_u) d\alpha = 0 \ , \qquad \text{as } f(\alpha\sigma_u) \text{ is an odd function} \tag{28}$$

$$a_1 = \frac{1}{\sigma_u \sqrt{2\pi}} \int_{-V_o}^{V_o} \exp\left(\frac{-\alpha^2}{2}\right) d\alpha = \frac{2}{\sqrt{\pi}} \int_o^{V_o(\sigma_u\sqrt{2})^{-1}} \exp(-\beta^2)\,d\beta$$

$$= \frac{2}{\sqrt{\pi}} \int_o^{65.29/\sigma_u} \exp(-\beta^2)\,d\beta \tag{29}$$

Let

$$\lambda_n(\alpha_o) = (-1)^n[2\pi(n!)]^{1/2} \left[\frac{d^{n-2}}{d\alpha^{n-2}} \exp\left(\frac{-\alpha^2}{2}\right)\right]_{\alpha=\alpha_o} \tag{30}$$

From equation (20)

$$\lambda_n(\alpha_o) = (-1)^n[2\pi(n!)]^{1/2} \left[\sqrt{(n - 2)!}\ (-1)^n \exp\left(\frac{-\alpha^2}{2}\right) \chi_{n-2}(\alpha)\right]_{\alpha=\alpha_o}$$

$$= [2\pi n(n - 1)]^{1/2} \exp\left(\frac{-\alpha_o^2}{2}\right) \chi_{n-2}(\alpha_o)$$

Using equation (A7) (in the appendix), with $\zeta_1 = \zeta_2 = \alpha_o$

$$\frac{d^2}{d\rho^2}\left[\sum_{n=2}^{\infty} \lambda_n^2(\alpha_o)\rho_u^n\right] = \sum_{n=2}^{\infty} [2\pi n(n - 1)]^{-1} \exp(-\alpha_o^2)\, \chi_{n-2}^2(\alpha_o)\, \frac{d^2}{d\rho^2}\, \rho^n$$

$$= \frac{\exp(-\alpha_o)}{2\pi} \sum_{m=o}^{\infty} \chi_m^2(\alpha_o)\rho_u^m$$

$$= \frac{\exp\left[\dfrac{-2\alpha_o^2(1 - \rho_u)}{2(1 - \rho_u^2)}\right]}{2\pi\sqrt{1 - \rho_u^2}} = \frac{1}{2\pi\sqrt{1 - \rho_u^2}} \exp\left(\frac{-\alpha_o^2}{1 + \rho_u}\right) \tag{31}$$

In this expression and those that follow $\rho_u(\tau)$ is written as ρ_u.

Let

$$\Lambda(\rho_u) = \sum_{n=2}^{\infty} a_n^2 \rho_u^n \tag{32}$$

From equations (26) and (30) it is seen that

$$\Lambda(\rho_u) = \sum_{n=2}^{\infty} [2\pi(n!)]^{-1} \left\{ \frac{d^{n-2}}{d\alpha^{n-2}} \left[\exp\left(\frac{-\alpha^2}{2}\right)\right]_{\alpha=\alpha_o} - \frac{d^{n-2}}{d\alpha^{n-2}} \left[\exp\left(\frac{-\alpha^2}{2}\right)\right]_{\alpha=-\alpha_o} \right\} \rho_u^n$$

$$= \sum_{n=2}^{\infty} [\lambda_n(\alpha_o) - \lambda_n(-\alpha_o)] \rho_u^n$$

Replacing α_o by $-\alpha_o$ in equation (30) shows that

$$\lambda_n(-\alpha_o) = (-1)^n \lambda_n(\alpha_o) \qquad n \geq 2$$

This permits $\Lambda(\rho_u)$ to be expressed as:

$$\Lambda(\rho_u) = \sum_{n=2}^{\infty} [\lambda_n(\alpha_o) - (-1)^n \lambda_n(\alpha_o)] \rho_u^n$$

$$= \sum_{n=2}^{\infty} [(\rho_u)^n - (-\rho_u)^n] \lambda_n(\alpha_o)$$

so that, on using equation (31)

$$\frac{d^2}{d\rho^2} \Lambda(\rho_u) = \frac{1}{2\pi\sqrt{1 - \rho_u^2}} \left[\exp\left(\frac{-\alpha_o^2}{1 + \rho_u}\right) - \exp\left(\frac{-\alpha_o^2}{1 - \rho_u}\right)\right]$$

from which:

$$\Lambda(\rho_u) = \int_0^{\rho_u} \left\{ \int_0^{\zeta} \frac{1}{2\pi\sqrt{1 - \lambda^2}} \left[\exp\left(\frac{-\alpha_o^2}{1 + \lambda}\right) - \exp\left(\frac{-\alpha_o^2}{1 - \lambda}\right)\right] d\lambda \right\} d\zeta$$

$$= \int_0^{\rho_u} \left\{ \int_\lambda^{\rho_u} \frac{1}{2\pi\sqrt{1 - \lambda^2}} \left[\exp\left(\frac{-\alpha_o^2}{1 + \lambda}\right) - \exp\left(\frac{-\alpha_o^2}{1 - \lambda}\right)\right] d\zeta \right\} d\lambda$$

or:

$$\Lambda(\rho_u) = \frac{1}{2\pi} \int_0^{\rho_u} \frac{(\rho_u - \lambda)}{\sqrt{1 - \lambda^2}} \left[\exp\left(\frac{-\alpha_o^2}{1 + \lambda}\right) - \exp\left(\frac{-\alpha_o^2}{1 - \lambda}\right)\right] d\lambda \quad (33)$$

This may be written as:

$$\Lambda(\rho_u) = \frac{1}{2\pi} \int_0^{\rho_u} \frac{(\rho_u - \lambda)}{\sqrt{1 - \lambda^2}} \left\{ \exp\left[\frac{-\alpha_o^2(1 - \lambda)}{1 - \lambda^2}\right] - \exp\left[\frac{-\alpha_o^2(1 + \lambda)}{1 - \lambda^2}\right] \right\} d\lambda$$

$$= \frac{1}{2\pi} \int_0^{\rho_u} \frac{(\rho_u - \lambda)}{\sqrt{1 - \lambda^2}} \exp\left(\frac{-\alpha_o^2}{1 - \lambda^2}\right) \left[\exp\left(\frac{\alpha_o^2\lambda}{1 - \lambda^2}\right) - \exp\left(\frac{-\alpha_o^2\lambda}{1 - \lambda^2}\right)\right] d\lambda$$

or

$$\Lambda(\rho_u) = \frac{1}{\pi} \int_0^{\rho_u} \frac{(\rho_u - \lambda)}{\sqrt{1 - \lambda^2}} \exp\left(\frac{-\alpha_o^2}{1 - \lambda^2}\right) \sinh\left(\frac{\alpha_o^2\lambda}{1 - \lambda^2}\right) d\lambda \quad (34)$$

Using equations (18), (28), (29), and (32) yields:

$$R_y(\tau) = \sigma_u^2 a_1^2 \rho_u + \frac{\sigma_u^2}{\pi} \int_0^{\rho_u} \frac{(\rho_u - \lambda)}{\sqrt{1 - \lambda^2}} \exp\left(\frac{-\alpha_o^2}{1 - \lambda^2}\right) \sinh\left(\frac{\alpha_o^2\lambda}{1 - \lambda^2}\right) d\lambda \quad (35)$$

To solve equation (35) as it stands would require that $\rho_u(\tau)$ be known beforehand. Although this is not the case, $\rho_u(\tau)$ and $R_y(\tau)$ are implicitly related through the closed-loop-system equations. A tractable expression for this interrelationship is obtained as follows:

Let the time history of the nonlinearity's input signal, $u(t)$, be approximated by a sequence of rectangular-shaped pulses each of width T_o and height $u(nT_o)$, above or below the abscissa, equal to the value of $u(t)$ at the beginning of the nT_oth interval. Approximations of this type have previously been used to analyze systems having no feedback.[3]

With this equivalence denoted by the symbol ~, the rectangular-pulse approximation for $u(t)$ can be expressed as:

$$u(t) \sim \sum_{n=-\infty}^{\infty} u\left(nT_o - \frac{1}{2} T_o\right) P(t - nT_o) \tag{36}$$

where

$$P(t) = \begin{cases} 1, & \text{for } |t| < T_o/2 \\ 0, & \text{elsewhere} \end{cases} \tag{37}$$

With ergodicity assumed, the rectangular pulse approximation of the autocorrelation function $\rho_u(\tau)$ at $\tau = mT_o$ is obtained as the following time average:

$$\rho_u(\tau) \sim \sigma_u^{-2}\left\langle \sum_{m=-\infty}^{\infty} u\left(nT_o - \frac{1}{2} T_o\right) P(t - nT_o)\, u\left(nT_o + mT_o - \frac{1}{2} T_o\right) P(t + \tau - mT_o - nT_o)\right\rangle$$

which may be written as:

$$\rho_u(\tau) \sim \sigma_u^{-2} \sum_{m=-\infty}^{\infty} \left\langle u\left(nT_o - \frac{1}{2}T_o\right) u\left(nT_o + mT_o - \frac{1}{2}T_o\right)\right\rangle P(\tau - mT_o)$$

$$= \sum_{m=-\infty}^{\infty} \rho_u(mT_o)P(\tau - mT_o)$$

With m_o chosen such that:

$$\rho_u(m_oT_o) \ll 1$$

and with

$$\rho_u(mT_o) = u_m \tag{38}$$

for convenience in the matrix methods that follow, one obtains:

$$\rho_u(\tau) \sim \sum_{m=-m_o}^{m_o} u_mP(\tau - mT_o) \tag{39}$$

The criterion for choosing T_o will be discussed later. By use of equations (18) and (38), $R_y(\tau)$ can be approximated as:

$$R_y(\tau) \sim \sigma_u^2 \sum_{n=o}^{\infty} \sum_{m=-m_o}^{m_o} a_n^2(u_m)^nP^n(\tau - mT_o)$$

Since

$$P^n(\tau - mT_o) = P(\tau - mT_o) \;, \qquad n \text{ a positive integer} \tag{40}$$

letting

$$y_m = \sigma_u^{-2}R_y(mT_o) = \sum_{n=o}^{\infty} a_n^2(u_m)^n \tag{41}$$

yields

$$R_y(\tau) \sim \sigma_u^2 \sum_{m=-m_o}^{m_o} y_m P(\tau - mT_o) \tag{42}$$

A tractable expression for the inverse Laplace transform of equation (25) is obtained in rectangular-pulse approximation form. Letting $*$ represent the convolation operation, m be a non-negative integer, and considering the interval $|(\tau - mT_o)| < T_o/2$, one obtains:

$$\begin{aligned} \mathbf{L}^{-1}\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\,\mathbf{L}R_y(\tau)\right]_{|\tau'|} &= \mathbf{L}^{-1}\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right] * R_y(\tau)_{|\tau'|} \\ &\sim \sigma_u^2\,\mathbf{L}^{-1}\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right] * \sum_{p=-m_o}^{m_o} y_p P(t - pT_o)_{|\tau'|} \\ &= \sigma_u^2 \sum_{p=m-m_o}^{m_o} \int_{(m-p-\frac{1}{2})T_o}^{(m-p+\frac{1}{2})T_o} \mathbf{L}^{-1}\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right] dt\; y_p P(|\tau| - mT_o)\ , \\ &\qquad \text{for } \tau = \tau',\ mT_o - \frac{T_o}{2} < |\tau'| < mT_o + \frac{T_o}{2} \end{aligned} \tag{43}$$

The domain of definition of equation (43) is expressed in terms of $|\tau'|$, rather than τ', because of the even symmetry of the inverse transformed quantities; namely, that:

$$R_y(\tau') = R_y(-\tau')$$

$$\mathbf{L}^{-1}\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right]\Bigg|_{\tau'} = \mathbf{L}^{-1}\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right]\Bigg|_{-\tau'}$$

Extending the domain of definition of equation (43) to $|\tau| < [m_o + (1/2)]T_o$ yields:

$$\mathbf{L}^{-1}\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\ \mathbf{L}R_y(\tau)\right]\Bigg|_{|\tau|<(m_o+\frac{1}{2})T_o}$$

$$\sim \sigma_u^2 \sum_{m=o}^{m_o} \sum_{p=m-m_o}^{m_o} y_p \int_{(m-p-\frac{1}{2})T_o}^{(m-p+\frac{1}{2})T_o} \mathbf{L}^{-1}\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right] dt\ P(|\tau| - mT) \tag{44}$$

The rectangular-pulse approximation for the inverse transform of the other terms to the right of the equal sign in equation (23) can also be expressed in this manner. This expression is:

$$\mathbf{L}^{-1}\left\{\mathbf{S}_u(s)\left[1 + \frac{(a_1 - \alpha_1)\mu(s)\mu(-s) - \alpha_1^2\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right]\right\}\Bigg|_{|\tau|<(m_o+\frac{1}{2})T_o}$$

$$\sim \sum_{m=o}^{m_o}\left\{\sigma_u^2 u_m + \sigma_u^2 \sum_{p=m-m_o}^{m_o} u_p \int_{(m-p-\frac{1}{2})T_o}^{(m-p+\frac{1}{2})T_o} \mathbf{L}^{-1}\left[\frac{(a_1 - \alpha_1)\mu(s)\mu(-s) - \alpha_1^2\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right] dt\right\} P(|\tau| - mT_o) \tag{45}$$

The inverse Laplace transform of the expression to the left of the equal sign in equation (25) can be written as

$$l(\tau) = \mathbf{L}^{-1}\left\{\left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right][K_o + K_1 s^4 \gamma(s)\gamma(-s)]\right\} * \delta(t)$$

The rectangular-pulse approximation for the dirac delta function, $\delta(t)$, is

$$\bar{\delta} = \frac{1}{T_o} P(t) \tag{46}$$

When this expression is used and the indicated convolution is performed, the rectangular-pulse approximation of $l(\tau)$ for $|\tau - mT_o| < 1/2$ is:

$$l_m = \frac{1}{T_o} \int_{(m-\frac{1}{2})T_o}^{(m+\frac{1}{2})T_o} \mathbf{L}^{-1} \left\{ \left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right] [K_o + K_1 s^4 \gamma(s)\gamma(-s)] \right\} d\tau \, P(\tau - mT_o) \tag{47}$$

The quantity has the property that:

$$l_m = l_{-m}$$

Using this and the symmetry of the other quantities in equation (25), letting:

$$\beta_m = \int_{(m-\frac{1}{2})T_o}^{(m+\frac{1}{2})T_o} \mathbf{L}^{-1} \left[\frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right] d\tau \tag{48}$$

$$\delta_m = \int_{(m-\frac{1}{2})T_o}^{(m+\frac{1}{2})T_o} \mathbf{L}^{-1} \left[\frac{\mu(s)\mu(-s)}{\xi(s)\xi(-s)}\right] d\tau \tag{49}$$

and equating the rectangular-pulse approximations for inverse Laplace transforms of the left- and right-hand sides of equation (25) yields:

$$\sum_{m=o}^{m_o} l_m P(|\tau| - mT_o) = \sigma_u^2 \sum_{m=o}^{m_o} \left\{ u_m + \sum_{p=m-m_o}^{m_o} \left[(a_1 - \alpha_1) u_{|p|} \delta_{|m-p|} \right.\right.$$
$$\left.\left. - \alpha_1^2 u_{|p|} \beta_{|m-p|} + y_{|p|} \beta_{|m-p|} \right] \right\} P(|\tau| - mT_o) \tag{50}$$

The convolution integral was used to derive the expression for l_m. This was done so that each l_m would be an averaged type quantity compatible with those indicated in equations (48) and (49).

By its definition, it is required that $\rho_u(0) = u_o = 1$. Using this, the symmetry property of the parameters, and equating coefficients of $P(|\tau| - mT_o)$ at $m = 0$ in equation (50) yields:

$$\frac{l_o}{\sigma_u^2} = 1 + (a_1 - \alpha_1)\delta_o + (y_o - \alpha_1^2)\beta_o + 2\sum_{p=1}^{m_o} [(a_1 - \alpha_1)u_p\delta_p + (y_p - \alpha_1^2 u_p)\beta_p] \tag{51}$$

Equation (50) can be written in a more convenient form for computative purposes. Let

$$\beta_{mp} = \begin{cases} \beta_{|m|}, & p = 0 \\ \beta_{|m-p|} + \beta_{|m+p|}, & 1 \leq |p| \leq m_o - m \\ \beta_{|m-p|}, & m_o - m + 1 \leq |p| \leq m_o \end{cases} \tag{52}$$

$$\delta_{mp} = \begin{cases} \delta_{|m|}, & p = 0 \\ \delta_{|m-p|} + \delta_{|m+p|}, & 1 \leq |p| \leq m_o - m \\ \delta_{|m-p|}, & m_o - m + 1 \leq |p| \leq m_o \end{cases} \tag{53}$$

Then equation (50) can be written

$$\sum_{m=o}^{m_o} l_m P(|\tau| - mT_o) = \sigma_u^2 \sum_{m=o}^{m_o} \left\{u_m + \sum_{p=o}^{m_o} [(a_1 - \alpha_1)\delta_{mp}u_p - (y_p - \alpha_1^2 u_p)\beta_{mp}]\right\} P(|\tau| - mT_o) \tag{54}$$

The matrix notation can be used to relate the coefficients of $P(|\tau| - mT_o)$ of this equation in a meaningful and concise manner. With L, U, and Y as $(m_o + 1)\times 1$ column matrices whose elements in the $(m + 1)$th row are l_m, u_m, and y_m, respectively; with B and Δ as

$(m_o + 1)\times(m_o + 1)$ square matrices, whose elements in the $(m + 1)$th row and $(n + 1)$th column are β_{mn} and δ_{mn}, respectively; and with I as the $(m_o + 1)\times(m_o + 1)$ unit matrix, the following relationship follows from equation (54):

$$L = \sigma_u^2[I + (a_1 - \alpha_1)\Delta - \alpha_1^2 B]U + \sigma_u^2 BY \tag{55}$$

Because of the symmetry of its autocorrelation functions, and the equations defining the components of B and Δ, equation (55) can be considered to be applicable to either positive or negative values of incremental time mT_o.

The relationship between $R_y(\tau)$ and $\rho_u(\tau)$ is obtained from equation (35). It will be convenient to restate this relationship in terms of u_m and y_m which have been defined since this equation was set down. The restatement is:

$$y_n = u_n(a_1)^2 + \frac{1}{\pi}\int_0^{u_n} \frac{(u_n - \lambda)}{\sqrt{1 - \lambda^2}} \exp\left[\frac{-V_o^2}{2\sigma_u^2(1 - \lambda^2)}\right] \sinh\left[\frac{\lambda V_o^2}{2\sigma_u^2(1 - \lambda^2)}\right] d\lambda$$

$$\text{where } V_o = 92.34, \text{ and } |n| \leq m_o \tag{56}$$

The computation process for finding U and, hence, the approximate expression for $\rho_u(\tau)$ will be discussed below. It is an iterative process in which successive values of U and the parameters from which it is computed are found, and one which terminates when these values do not change significantly. To index the values found during a particular iteration, the ith iteration, a subscript (i) is added to the symbol of the parameter being computed (e.g., $\sigma_{u(i)}$ is the value of σ_u found on the ith iteration). Except for finding the initial set of values $(i = 0)$, the same iteration process is used until the computed values converge.

To initiate the computation process, the nonlinearity is replaced by a linear gain $a_{1(o)}$ such that $R_{y(o)}(\tau) = a_{1(o)}^2 R_{u(o)}(\tau)$. With this substitution, and with $\alpha_1 = a_1 = a_{1(o)}$ and $s = j\omega$, equation (25) becomes

$$\mathbf{S}_{u(o)}(j\omega) = \frac{\eta(j\omega)\eta(-j\omega)[K_o + K_1\omega^4\gamma(j\omega)\gamma(-j\omega)]}{[\xi(j\omega)\xi(-j\omega)]_{\alpha_1=a_{1(o)}}\gamma(j\omega)\gamma(-j\omega)}$$

$$= \frac{1.274\times10^4(\omega^4 + 33.59\omega^2 + 0.2278)(\omega^6 + 0.1858\omega^4 + 8.878)}{\left(\omega^2[\omega^4 - 12.01\omega^2 + 33.01a_{1(o)}]^2 + \{4.915\omega^4 - [29.14a_{1(o)} + 4.448]\omega^2 + 13.91a_{1(o)}\}^2\right)(\omega^2 + 0.1858)} \tag{57}$$

In this equation s has been set equal to $j\omega$. As will be seen, this will enable the expression to be used in evaluating T_o as well as σ_u. From its definition, σ_u is evaluated as:

$$\sigma_{u(o)} = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{S}_{u(o)}(j\omega)d\omega\right]^{1/2} \tag{58}$$

From equation (29):

$$a_{1(o)} = \int_0^{65.29/\sigma_{u(o)}} \exp(-\beta^2)d\beta \tag{59}$$

Values for $\sigma_{u(o)}$ and $a_{1(o)}$ are obtained by simultaneously solving equations (58) and (59).

By the evaluation of equation (57) T_o is computed as a function of ω. Following a region of rapid change near the origin, $\mathbf{S}_{u(o)}(j\omega)$ decreases monotonically with $|\omega|$ in a well-behaved manner. After a certain value of $|\omega|$, $|\omega_o|$, the value of $\mathbf{S}_{u(o)}(j\omega)$ can be considered to be insignificant as compared with that at the beginning of the well-behaved region. Since $\mathbf{S}_{u(o)}(j\omega) = \mathbf{S}_{u(o)}(-j\omega)$, this behavior need not be noted for both positive and negative values of ω. On the assumption that $\mathbf{S}_{u(o)}(j\omega)$ is bandlimited to $\pm|\omega_o|$, Nyquist's sampling theorem is used to compute T_o as:

$$T_o = \frac{1}{2|\omega_o|} \tag{60}$$

where ω_o is the frequency at which the value of $\mathbf{S}_{u(o)}(j\omega)$ is considered to be insignificant.

Next, $R_{u(o)}(\tau)$ is evaluated as a function of τ. It will be found that its value decreases monotonically with $|\tau|$ in a well-behaved manner in a region away from the origin. At an integer, m_o, the value of $R_{u(o)}(|m_o|T_o)$ can be considered to be insignificant as compared to its value at the beginning of the well-behaved region. Because $R_{u(o)}(mT_o) = R_{u(o)}(-mT_o)$, this behavior need be noted for either positive or negative values of mT_o but not for both. The m_o found by this process is the m_o referred to above.

The values of T_o, m_o, and $a_{1(o)}$ found above are used to compute a set of l_m from equation (54) for $0 \leq m \leq m_o$, forming the vector $[L]_{a_{1(o)}}$. Since

$$Y_{(o)} = a^2_{1(o)}U_{(o)} \tag{61}$$

setting $a_1 = a_{1(o)} = \alpha_1$ in equation (55) yields

$$U_{(o)} = \frac{[L]_{a_{1(o)}}}{\sigma^2_{u(o)}} \tag{62}$$

Before continuing the computation process, it is necessary to change α_1 by a small amount from its value of $a_{1(o)}$. If this were not done the effect of the nonlinearity would be approximated by only the linear gain $a_{1(o)}$. This can be seen from equation (54). The change in α_1 will affect the waveshape of $\mathbf{S}_{u(o)}(j\omega)$ and its bandwidth value $|\omega_o|$. It should be limited to a value which does not significantly alter the value of ω_o and, hence, the values of T_o and m_o found from it.

It may be assumed that if a change in α_1 is accompanied by a relatively insignificant change in the dominant roots of the characteristic equation, ω_o will change by a comparatively small amount. Since the roots of the characteristic equation which change with α_1 are those contributed by $\xi(s)$, the effect that a change in α_1 has on ω_o can be gaged qualitatively by plotting the roots of $\xi(s)$ in the complex s plane as a function of α_1.

Equation (55), with equation (56) substituted into it, reveals that σ_u is proportional to the difference between α_1 and a_1. Equation (29), on the other hand, shows that increases or decreases in a_1 are reflected oppositely in changes of σ_u. This indicates that the value of α_1 now selected may be chosen to produce a self-correcting type action between the values of a_1 and σ_u found in the subsequent iterations; and, furthermore, that its value may be chosen to speed up the convergence of the iteration process. During the iteration, α_1 could be changed continually; but since L, Δ, and B must be recomputed each time α_1 is changed, this does not seem practical. Before α_1 is changed, the need for the alteration should be carefully ascertained.

As an example of the use of the technique for determining α_1, the situation pertinent to the particular problem being studied will be described. For $\alpha_1 = a_{1(o)}$ the real part of the dominant root of $\xi(s)$, in equation (23), is equal to 0.6927. Decreasing α_1 causes its value to drop quite rapidly; whereas, increasing α_1 by 1.0 percent causes it to increase by about 0.64 percent to its maximum value. A further increase of 6.5 percent in α_1 causes the real part of dominant root to decrease by about the same percentage. Thus, the real part of dominant root increases slightly and then gradually decreases as α_1 is increased from its value of $a_{1(o)}$. This behavior suggests that the self-correcting type action described above would be more likely to occur if α_1 were chosen so that the real part of the dominant root were to lie on the locus which is on the other side of the maximum to that corresponding to $\alpha_1 = a_{1(o)}$. The new value of α_1 is, therefore, taken to be 0.4700, an increase of 1.0 percent + 6.5 percent from $a_{1(o)} = 0.4400$.

This value of α_1 is used to evaluate L, B, and Δ. Unless a change in its value is required for aiding the convergence of the computation process, the value of α_1 and those of L, B, and Δ will not be altered for the remainder of the process; consequently, no (i) subscripts are added to the notation of these parameters.

The general iteration process will now be described. Under the assumption that $a_{1(i)}$, $\sigma_u(i)$, $U(i)$, and $Y(i)$ are known for $i \geq 0$, the $(i + 1)$th set of values is computed as follows:

(1) $\sigma_{u(i+1)}$ is computed from either equation (51) or the first row of equation (55) with $a_1 = a_{1(i)}$, $U = U(i)$, and $Y = Y(i)$.

(2) $a_{1(i)}$ is computed from equation (29) with $\sigma_u = \sigma_u(i+1)$.

(3) $U_{(i+1)}$ is computed from equation (55) with $\sigma_u = \sigma_u(i+1)$, $a_1 = a_{1(i+1)}$, and $Y = Y(i)$, as follows:

$$U_{(i+1)} = \left[\frac{L}{\sigma^2_{u(i+1)}} - BY_{(i)}\right]\left[I + [a_{1(i+1)} - \alpha_1]\Delta - a^2_{1(i+1)}B\right]^{-1} \tag{63}$$

(4) $Y_{(i+1)}$ is computed from equation (56) with $\sigma_u = \sigma_{u(i+1)}$, $a_1 = a_{1(i+1)}$, and $U = U_{(i+1)}$.

Due to the size of the matrix, a digital computer should be used to perform the matrix inversion required in step (3) of the iteration process. With α_1 constant, the program required to recalculate the inverse matrix between iterations should be relatively simple, since the only change in the matrix is that due to a_1, which enters as a linear coefficient of Δ.

The order of computation has been chosen to insure that U will be computed from the most recently computed values of σ_u and a_1. The process also insures that $u_{o(i)} = 1$ for all values of i. This is necessary since U represents a normalized autocorrelation function.

Now that the rectangular-pulse approximation for $\rho_u(\tau)$ has been obtained, the next step is to find a similar expression for $R_v(\tau)$. Let the rectangular-pulse approximation for $R_v(\tau)$ be given by:

$$R_v(\tau) = \sigma_u{}^2 \sum_{m=-m_o}^{m_o} v_m P(\tau - mT_o) \tag{64}$$

Since:

$$\sigma_u{}^2\rho_u(\tau) = [\mathbf{L}^{-1}H(s)H(-s)] * R_v(\tau) = \left[\mathbf{L}^{-1}\frac{\eta(s)\eta(-s)}{\nu(s)\nu(-s)}\right] * R_v(\tau)$$

Using equations (39) and (64) yields:

$$\sigma_u^2 \sum_{m=o}^{m_o} u_m P(|\tau| - mT_o)$$

$$= \sigma_u^2 \sum_{m=o}^{m_o} \sum_{p=m-m_o}^{m_o} v_p \int_{(m-p-\frac{1}{2})T_o}^{(m-p+\frac{1}{2})T_o} \frac{\eta(s)\eta(-s)}{\nu(s)\nu(-s)} dt P(|\tau| - mT_o) \tag{65}$$

If $\xi(s)$ is replaced by $\nu(s)$ and $\xi(-s)$ by $\nu(-s)$, ψ_{mp} is defined analogously to β_{mp}. With this notation, equation (65) becomes

$$\sum_{m=o}^{m_o} u_m P(|\tau| - mT_o) = \sum_{m=o}^{m_o} \sum_{p=o}^{m_o} \psi_{mp} v_p P(|\tau| - mT_o) \tag{66}$$

With u and V as $(m_o + 1)\times 1$ column matrices whose $(m + 1)$th row elements are u_m and v_m, respectively, and with Ψ as a $(m_o \times 1)\times(m_o \times 1)$ square matrix whose element in the $(m + 1)$th row and $(n + 1)$th column is ψ_{mn}, equation (66) can be written as:

$$U = \Psi V \tag{67}$$

or, with the usual notation for the inverse of a matrix, as:

$$V = \Psi^{-1} U \tag{68}$$

The external input to the node preceding $H(s)$, $n(t)$, represents zero mean white Gaussian noise and, because of this, is uncorrelated with $e(t)$, the other input at this node. This means that the expected value

$$\langle e(t)n(t)\rangle = \langle e(t)\rangle \langle n(t)\rangle = 0 \tag{69}$$

so that

$$R_v(\tau) = R_m(\tau) + R_e(\tau) \tag{70}$$

Using equations (4), (13d), and (46) the rectangular-pulse approximation for $R_n(\tau)$ is obtained from:

$$R_n(\tau) = A_1{}^2\delta(\tau) = K_1\delta(\tau)$$

as

$$R_n(\tau) \sim \frac{K_1}{T_o} P(\tau) \tag{71}$$

and the rectangular-pulse approximation of the root-mean-square error signal, $R_e(0)$, is obtained from:

$$R_e(0) = R_v(0) - R_n(0)$$

as

$$R_e(0) \sim \sigma_u{}^2 v_o - \frac{K_1}{T_o} \tag{72}$$

This is the end result sought in the study. It will be noted that only the first row element of V, v_o, was required to obtain it.

Because of the low-pass characteristics of the filter, $H(s)$, the significant bandwidth of $\mathbf{S}_{u(j\omega)}$ will be less than that of $\mathbf{S}_{v(j\omega)}$. This effect is reflected oppositely in the significant domains of definition of the autocorrelation functions $R_u(\tau)$ and $R_v(\tau)$. As compared with that of $R_u(\tau)$, the significant domain of definition of $R_v(\tau)$ will be decreased. This means that the value of m_oT_o determined for $R_u(\tau)$ may be applied to $R_v(\tau)$ without deteriorating the accuracy of the results.

ANALYSIS OF A MORE GENERAL CLASS OF NONLINEAR SYSTEMS

In the interception problem the relationship between the input and output autocorrelation functions could be expressed in closed form. In the general situation, where this need not be the case, a slight change in the procedure may be required. The analysis need not be restricted to systems whose block diagrams can be reduced to that indicated in figure 2. By simply redefining the quantities it can be extended to a much broader class.

The purpose of this section will be to extend the analysis to more general classes of nonlinearities and systems. When a symbol here has the same meaning as that used in the previous analysis, the defining equation above will be referred to. It will consider systems whose input is a stationary real random process subject to additive zero mean white Gaussian noise. The system configuration should be physically realizable, can be reduced to the block diagram of figure 3, and meets a minimum phase condition which is indicated below.

The block diagram of the general type system is indicated in figure 3. The quantities which indicate the transforms of the analog signals are equivalent to those of figure 2.

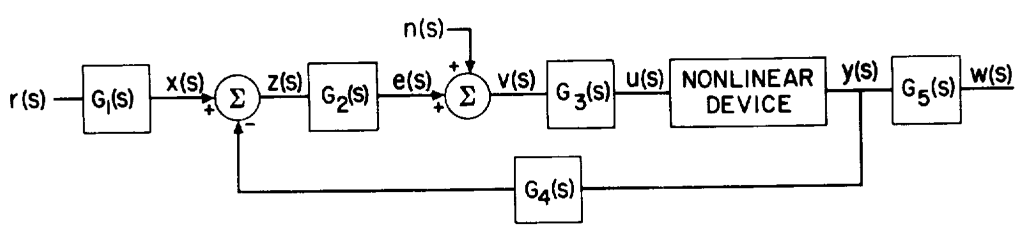

Figure 3. - Block diagram of a general class of systems.

The following relationships exist among the quantities indicated in figure 3.

$$G_3(s)[G_1(s)G_2(s)r(s) + n(s)] = u(s) + G_2(s)G_3(s)G_4(s)y(s) \quad (73)$$

The relationship between the quantities at input and output of the nonlinearity, f(u), are indicated in equations (17) through (22), inclusive. Using these relationships, and the fact that e(t) and n(t) are uncorrelated yields:

$$\begin{aligned} G_3(s)G_3(-s)[G_1(s)G_1(-s)G_2(s)G_2(-s)\,\mathbf{S}_r(s) + \mathbf{S}_n(s)] \\ = \mathbf{S}_u(s) + a_1[G_2(s)G_3(s)G_4(s) + G_2(-s)G_3(-s)G_4(-s)]\,\mathbf{S}_u(s) \\ + G_2(s)G_2(-s)G_3(s)G_3(-s)G_4(s)G_4(-s)\,\mathbf{S}_y(s) \end{aligned} \quad (74)$$

Defining

$$G_n(s) = \frac{p_n(s)}{q_n(s)} \quad (75)$$

$$\eta(s) = p_2(s)p_3(s)p_4(s) \quad (76)$$

$$\nu(s) = q_2(s)q_3(s)q_4(s) \quad (77)$$

$$\xi(s) = \nu(s) + a_1\eta(s) \quad (78)$$

$$\mu(s) = \left[s^{-m_1}\prod_{i=1}^{m_2}(s^2 + d_i^2)^{-1}\right]\left\{\eta(s)\nu(-s) + \eta(-s)\nu(s)\right\}^+ \quad (79)$$

where m_1 and m_2 are integers, d_i are real constants, and the meaning of the symbol $\{\ \}^+$ is indicated below equation (24).

$$\mathbf{S}_{Ko}(s) = \left[\frac{p_3(s)p_3(-s)}{\xi(s)\xi(-s)}\right] G_1(s)G_1(-s)G_2(s)G_2(-s)\,\mathbf{S}_r(s) \quad (80)$$

$$\mathbf{S}_{K_1}(s) = \left[\frac{p_3(s)p_3(-s)}{\xi(s)\xi(-s)}\right]\mathbf{S}_n(s) \tag{81}$$

Analogous to equation (25), equation (74) can be written as:

$$\mathbf{S}_{K_0}(s) + \mathbf{S}_{K_1}(s)$$

$$= \left[1 + \frac{(a_1 - \alpha_1)s^{2m_1}\prod_{i=1}^{m_2}(s^2 + d_i{}^2)^2\mu(s)\mu(-s) - \alpha_1{}^2\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\right]\mathbf{S}_u(s)$$

$$+ \frac{\eta(s)\eta(-s)}{\xi(s)\xi(-s)}\,\mathbf{S}_y(s) \tag{82}$$

The zeros of $\eta(s)$ and $\nu(s)$ which occur on the imaginary axis in the complex s plane were eliminated from the definition of $\mu(s)$ in equation (79). Those which occur at origin were eliminated by the s^{-m_1} factor, and those occurring in conjugate pairs were eliminated by the $\prod_{i=1}^{m_2}(s^2 + d_i{}^2)^{-1}$ factor. The method of this analysis is restricted to systems for which the resulting $\mu(s)$ can be formed so as to have all its zeros in the left-half side of the complex s plane. It is also restricted to stable systems and is, hence, restricted to systems where $\xi(s)$ also has this property. The restriction amounts to requiring that $\mu(s)/\xi(s)$ be a minimum phase function, one having all its poles and zeros in the left-hand side of the complex s plane.

Except for equations (45), (47), (49), (56), (57), and (59), equations (36) through (63), inclusive, are applicable to the present analysis.

Equation (59) is replaced by equation (19) with $n = 1(0)$ and $\sigma_u = \sigma_u(0)$. Equations (47), (49), and (57) are replaced by equations (83), (84), and (85), respectively, where:

$$l_m = \frac{1}{T_o} \int_{(m-\frac{1}{2})T_o}^{(m+\frac{1}{2})T_o} \mathbf{L}^{-1}[\mathbf{S}_{Ko}(s) + \mathbf{S}_{K1}(s)]d\tau P(\tau - mT_o) \qquad (83)$$

$$\delta_m = \int_{(m-\frac{1}{2})T_o}^{(m+\frac{1}{2})T_o} \mathbf{L}^{-1}\left[\frac{s^{2m_1}\prod_{i=1}^{m_2}(s^2 + d_i)^2\mu(s)\mu(-s)}{\xi(s)\xi(-s)}\right]d\tau \qquad (84)$$

$$\mathbf{S}_{u(o)}(j\omega) = [\mathbf{S}_{K_o}(s) + \mathbf{S}_{K_1}(s)]\Big|_{\substack{s=j\omega \\ \alpha=a_{1(o)}}} \qquad (85)$$

Equation (45) must be modified. The quotient within the brackets on both sides of this equation must be replaced by the quotient within the bracket of equation (82).

When it is possible to obtain a closed-form relationship between the input and output autocorrelation functions of the nonlinear device of the system, this relationship, expressed in rectangular-pulse approximation form, is substituted for equation (56) of the computational process. When it is not possible to find such a relationship, the power series relationship indicated by equation (41) is used instead.

If the indicated substitutions are used, the rectangular-pulse approximation expression for $R_u(\tau)$ is found by the same iterative numerical method discussed relative to the interception problem. The possible necessity of relating $R_u(\tau)$ and $R_y(\tau)$ by the power series expression of equation (41) presents no difficulty in this method. The expression for $R_y(\tau)$ in a given interval of time, mT_o, is related to that of $R_u(\tau)$ in the same interval of time by this equation. Because of the properties of the pulse-approximation method, the value of $R_u(\tau)$ in other intervals of time do not enter into this computation, thus permitting $R_y(\tau)$ to be calculated as a set of scalar equations. $R_u(\tau)$ is found from equation (55) rather than from the inverted relationship indicated by this set of equations.

The method for finding the rectangular-pulse approximation for $R_v(\tau)$ here is essentially the same as that indicated previously. Equation (64) is directly applicable, and the method proceeds as follows: Let

$$d_m = \int_{(m-\frac{1}{2})T_o}^{(m+\frac{1}{2})T_o} \mathbf{L}^{-1}[G_3(s)G_3(-s)]dt \tag{86}$$

$$d_{mp} = \begin{cases} d_{|m|}\ , & p = 0 \\ d_{|m-p|} + d_{|m+p|}\ , & 1 \leq |p| \leq m_o - m \\ d_{|m-p|}\ , & m_o - m + 1 \leq |p| \leq m_o \end{cases} \tag{87}$$

With D as a $(m_o + 1)\times(m_o + 1)$ square matrix, whose element in the $(m + 1)$th row and $(n + 1)$th column is d_{mn}, and V as a $(m_o + 1)\times 1$ column matrix, whose element in the $(m + 1)$th row is v_m, the relationship between the rectangular-pulse approximations of the autocorrelation functions at the output and input of the system component designated as $G_3(s)$ is, in matrix notation:

$$U = DV$$

from which, using the usual notation for the inverse of a matrix, one obtains:

$$V = D^{-1}U \tag{88}$$

The method for finding the root-mean-square value of the error, $e(t)$, from v_o is the same as that indicated for the interception problem. Since the noise is white, the rectangular-pulse approximation of its inverse Laplace transform is a constant, $(\sigma_n)^2/T_o$. It is obtained as follows:

$$\mathbf{L}^{-1}\mathbf{S}_n(s) = (\sigma_n)^2\delta(\tau) \sim \frac{(\sigma_n)^2}{T_o} \tag{89}$$

If the noise is uncorrelated with the error signal and has zero mean, the rectangular-pulse approximation for error is:

$$R_e(0) = \sigma_u{}^2 v_o - \frac{(\sigma_n)^2}{T_o} \tag{90}$$

A particular example of extending the method of analysis to a more general type situation will be indicated before concluding this section of the study. This example is that of a system having the same configuration as the interception problem but with the limiter replaced by a nonlinearity whose static response is monotonically increasing, has odd symmetry, and can be represented by a number of straight-line segments. With $(u)_m$ denoting the ordinate at which its slope changes, the static response of the nonlinearity can be represented as:

$$f[u(t)] = \sum_{n=1}^{\infty} k_n' u(t)[Q_n(u) - Q_{n-1}(u)] = \sum_{n=1}^{N} (k_n' - k_{n+1}')u(t)Q_n(u)$$

or

$$f[u(t)] = \sum_{n=1}^{N} k_n u(t) Q_n(u) \tag{91}$$

where

$$f[u(t)] = -f[-u(t)]$$

$$f(0) = (u)_o = 0$$

$$|f[u(t)]| = f[(u)_N] , \qquad \text{for } |u(t)| \geq (u)_N$$

and with

$$Q_n(u) = \begin{cases} 1 , & \text{for } |u(t)| < (u)_n \\ 0 , & \text{for } |u(t)| \geq (u)_n \end{cases} \tag{92}$$

$$k_n' = \frac{f[(u)_n] - f[(u)_{n-1}]}{(u)_n - (u)_{n-1}} \tag{93}$$

$$k_n = k_n' - k_{n+1}' \tag{94}$$

By use of equation (91) the a_n of equation (18) can be found like those of equations (26), (28), and (29) as follows:

$$a_n = \frac{(-1)^{n+1}}{\sqrt{2\pi(n!)}} \sum_{m=1}^{N} k_m \left[\frac{d^{n-2}}{d\alpha^{n-2}} \exp\left(\frac{-\alpha^2}{2}\right)\right]_{\alpha=-(u)_m/\sigma_u}^{\alpha=(u)_m/\sigma_u} , \qquad n \geq 2 \tag{95}$$

$$a_o = \frac{1}{\sigma_u\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha\sigma_u)d\alpha = 0 , \qquad \text{as } f(u) \text{ is an odd function} \tag{96}$$

$$a_1 = \frac{2}{\sqrt{\pi}} \sum_{m=1}^{N} k_m \int_0^{(u)_m(\sigma_u\sqrt{2})^{-1}} \exp(-\beta^2)d\beta \tag{97}$$

Using these equations, one obtains instead of equation (56) the following:

$$y_n = u_n(a_1)^2$$

$$+ \frac{1}{\pi} \int_0^{u_n} \frac{(u_n - \lambda)}{\sqrt{1 - \lambda^2}} \sum_{m=1}^{N} k_m \exp\left\{\frac{-[(u)_m]^2}{2\sigma_u^2(1 - \lambda^2)}\right\} \sinh\left\{\frac{\lambda[(u)_m]^2}{2\sigma_u^2(1 - \lambda^2)}\right\} d\lambda \tag{98}$$

Upon substitution of these expressions, the analysis proceeds as in the interception problem.

CONCLUDING REMARKS

The restrictions imposed on the input statistics and system configuration are sufficiently broad so that the method can be applied to a wide class of problems. The method does not require that a filter designed to minimize the root-mean-square value of the signal at the error node be present or that this error signal be a measure of the difference between the input and output of the system; but the results are most meaningful under these conditions.

The rectangular-pulse approximation which is used permits the autocorrelation response at the input of the nonlinear device to be multiplied by itself any number of times. The quantity obtained maintains the characteristic that its value at any interval of time depends only on the value of the input autocorrelation function in that interval of time, and not on any other interval of time. This, together with Mehler's expansion, provides a tractable method for finding the output autocorrelation function of the nonlinear device as a power series in terms of its normalized input autocorrelation function. The flexibility of the technique is enhanced by the methodology of the iterative process, which permits the input autocorrelation function to be derived from a relatively simple matrix expression, rather than from the inverted power series obtained from the Mehler expansion. This process is mechanized for a digital computer so that answers may be obtained quickly.

The method provides a tool for obtaining numerical and analytic expressions for the autocorrelation functions at various points in the system, as well as the root-mean-square value of the signal at the error node. As such, it provides an instrument for viewing what effect changes in the configuration of the component, referred to as Wiener filter, have on these quantities. Since the interrelationships between the parameter of the filter and these quantities are implicit in this method, the modus operandi of the investigation would be experimental. Computer techniques would enable the filter parameters to be varied; the changes in the error and the waveshapes of the autocorrelation functions would then be noted. A mathematical model for the process would at first be obtained by means of perturbation techniques. The changes in the filter parameters would be small and the interrelationships would be derived from a quasi-linear type analysis.

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APPENDIX

The autocorrelation function, $R_y(t_1,t_2)$, at the output of a nonlinear device can be expressed as a function of its static response, $f(u)$, and the joint probability density, $p(u_1,u_2)$, of its input as follows:

$$R_y(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u_1)f(u_2)p(u_1,u_2)du_1\, du_2 \tag{A1}$$

where

$$u_1 = u(t_1) \text{ and } u_2 = u(t_2)$$

When the input statistics are Gaussian with standard deviation σ_u and normalized autocorrelation function ρ_u, equation (1) may be written as:

$$R_y(t_1,t_2)$$
$$= \frac{1}{2\pi\sigma_u^2\sqrt{1-\rho_u^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{-\infty} f(u_1)f(u_2)\ \exp\left[\frac{-(u_1^2+u_2^2-2\rho_u u_1 u_2)}{2\sigma_u^2(1-\rho_u^2)}\right] du_1\, du_2 \tag{A2}$$

Mehler's expansion of the exponential in this equation may be used to express $R_y(t_1,t_2)$ in terms of a power series of the normalized input autocorrelation function, ρ_u. The expression for this expansion will now be derived.

The nth Hermite polynomial is defined as:

$$H_n(\varphi) = (-1)^n \exp(\varphi^2) \frac{d^n}{d\varphi^n} \exp(-\varphi^2) \tag{A3}$$

Since

$$\exp(-\varphi^2) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\alpha^2 + 2j\alpha\varphi)d\alpha \tag{A4}$$

where

$$j = \sqrt{-1}$$

equation (A3) becomes:

$$H_n(\varphi) = (-1)^n(\pi)^{-1/2} \exp(\varphi^2) \int_{-\infty}^{\infty} (2j\alpha)^n \exp(-\alpha^2 + 2j\alpha\varphi)d\alpha \tag{A5}$$

as

$$\exp(-2\rho_u\alpha\beta) = \sum_{n=o}^{\infty} \frac{(-2\rho_u\alpha\beta)^n}{n!}$$

equation (A5) yields:

$$\sum_{n=o}^{\infty} \frac{H_n(\varphi_1)H_n(\varphi_2)\rho_u{}^n}{2^n n!} = \sum_{n=o}^{\infty} \frac{\exp(\varphi_1{}^2 + \varphi_2{}^2)}{\pi n!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (2j)^{2n}\rho_u{}^n(\alpha\beta)^n 2^{-n} \exp(-\alpha^2$$

$$+ 2j\alpha\varphi_1 - \beta^2 + 2j\beta\varphi_2)d\alpha\ d\beta$$

$$= \sum_{n=o}^{\infty} (\pi n!)^{-1} \exp(\varphi_1{}^2 + \varphi_2{}^2) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (-2\rho_u\alpha\beta)^n \exp(-\alpha^2$$

$$- \beta^2 + 2j\alpha\varphi_1 + 2j\beta\varphi_2)d\alpha\ d\beta$$

$$= \pi^{-1} \exp(\varphi_1{}^2 + \varphi_2{}^2) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-\alpha^2 - \beta^2 + 2j\alpha\varphi_1$$

$$+ 2j\beta\varphi_2 - 2\rho_u\alpha\beta)d\alpha\ d\beta$$

Using equation (A4):

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\beta^2 + 2j\beta\varphi_2 - 2\rho_u\alpha\beta)d\beta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\beta^2 + 2j\beta(\varphi_2 + j\alpha\beta)d\beta$$

$$= \exp[-(\varphi_2 + j\alpha\rho_u)^2]$$

Substituting this above and then using equation (A4) again yields:

$$\sum_{n=0}^{\infty} \frac{H_n(\varphi_1)H_n(\varphi_2)\rho_u{}^n}{2^n n!} = \pi^{-1/2} \exp(\varphi_1{}^2 + \varphi_2{}^2) \int_{-\infty}^{\infty} \exp[-(\varphi_2 + j\alpha\rho_u)^2 - \alpha^2$$

$$+ 2j\alpha\varphi_1]d\alpha$$

$$= \pi^{-1/2} \exp(\varphi_1{}^2) \int_{-\infty}^{\infty} \exp[-\alpha^2(1 - \rho_u{}^2)$$

$$+ 2j\alpha(\varphi_1 - \rho_u\varphi_2)]d\alpha$$

$$= [\pi(1 - \rho_u{}^2)]^{-1/2} \exp(\varphi_1{}^2) \int_{-\infty}^{\infty} \exp[-\alpha^2$$

$$+ 2j\alpha(\varphi_1 - \rho_u\varphi_2)(1 - \rho_u{}^2)^{-1/2}]du$$

$$= (1 - \rho_u{}^2)^{-1/2} \exp(\varphi_1{}^2) \exp[-(\varphi_1 - \rho_u\varphi_2)^2(1 - \rho_u{}^2)^{-1}]$$

$$= (1 - \rho_u{}^2)^{-1/2} \exp(\varphi_1{}^2 + \varphi_2{}^2) \exp[-(\varphi_1{}^2 + \varphi_2{}^2$$

$$- 2\rho_u\varphi_1\varphi_2)(1 - \rho_u{}^2)^{-1}]$$

from which is obtained:

$$(1 - \rho_u{}^2)^{-1/2} \exp[-(\varphi_1{}^2 + \varphi_2{}^2 - 2\rho_u\varphi_1\varphi_2)(1 - \rho_u{}^2)^{-1}]$$

$$= \exp(-\varphi_1{}^2 - \varphi_2{}^2) \sum_{n=0}^{\infty} \frac{H_n(\varphi_1)H_n(\varphi_2)\rho_u{}^n}{2^n n!} \qquad \text{(A6)}$$

Letting

$$\varphi_1 = 2^{-1/2}\zeta_1$$

and

$$\varphi_2 = 2^{-1/2}\zeta_2$$

and introducing the Hermite polynomials based on $\exp(-\zeta^2/2)$ and defined by

$$\chi_n(\zeta) = (-1)^n(n!)^{-1/2} \exp\left(\frac{\zeta^2}{2}\right) \frac{d^n}{d\zeta^n} \exp\left(\frac{-\zeta^2}{2}\right)$$

$$= (n!2^n)^{-1/2} H_n(2^{-1/2}\zeta)$$

into equation (A6) yields

$$(1 - \rho_u^2)^{-1/2} \exp\left[\frac{-\zeta_1^2 - \zeta_2^2 + 2\rho_u\zeta_1\zeta_2}{2(1 - \rho_u^2)}\right]$$

$$= \exp\left[\frac{-(\zeta_1^2 + \zeta_2^2)}{2}\right] \sum_{n=0}^{\infty} \chi_n(\zeta_1)\chi_n(\zeta_2)\rho_u^n \tag{A7}$$

Letting

$$\frac{u_1}{\sigma_u} = \zeta_1$$

and

$$\frac{u_2}{\sigma_u} = \zeta_2$$

in equation (A2) and substituting equation (A7) into the resulting expression yields

$$R_y(t_1,t_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\sigma_u\zeta_1)f(\sigma_u\zeta_2) \exp\left[\frac{-(\zeta_1^2 + \zeta_2^2)}{2}\right] \sum_{n=o}^{\infty} \chi_m(\zeta_1)\, \chi_m(\zeta_2)\rho_u^n \, d\zeta_1 \, d\zeta_2$$

$$= \frac{1}{2\pi} \sum_{n=o}^{\infty} \rho_u^n \int_{-\infty}^{\infty} f(\sigma_u\zeta_1)\, \chi_n(\zeta_1) \exp\left(\frac{-\zeta_1^2}{2}\right) d\zeta_1 \int_{-\infty}^{\infty} f(\sigma_u\zeta_2)\, \chi_n(\zeta_2) \exp\left(\frac{-\zeta_2^2}{2}\right) d\zeta_2 \tag{A8}$$

or

$$R_y(t_1,t_2) = \sigma_u^2 \sum_{n=o}^{\infty} a_n^2\rho_u^n \tag{A9}$$

where

$$a_n = \frac{1}{\sigma_u\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\sigma_u\zeta)\, \chi_n(\zeta) \exp\left(\frac{-\zeta^2}{2}\right) d\zeta \tag{A10}$$

when the input statistics are stationary, equation (A9) may be written as:

$$R_y(\tau) = \sigma_u^2 \sum_{n=o}^{\infty} a_n^2\rho_u^n \tag{A11}$$

where

$$\tau = t_1 - t_2$$

Replacing $f(u_1)$ by u_1 in equation (A2) and then substituting equation (A7), $u_1/\sigma_u = \zeta_1$, and $u_2/\sigma_u = \zeta_2$ into the resulting expression yields

$$R_{uy}(t_1,t_2) = R_{yu}(t_1,t_2) = \frac{\sigma_u}{2\pi} \sum_{n=o}^{\infty} \rho_u^n \int_{-\infty}^{\infty} \zeta_1\, \chi_m(\zeta_1) \exp\left(\frac{-\zeta_1^2}{2}\right) d\zeta_1 \int_{-\infty}^{\infty} f(\sigma_u\zeta_2)\, \chi_n(\zeta_2) \exp\left(\frac{-\zeta_2^2}{2}\right) d\zeta_2 \tag{A12}$$

Because of the orthonormal property of $\chi_n(\zeta)$:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \zeta \chi_n(\zeta) \exp\left(\frac{-\zeta^2}{2}\right) d\zeta = \begin{cases} 1, & \text{for } n = 1 \\ 0, & \text{for } n \neq 1 \end{cases} \tag{A13}$$

and with the use of this and equation (A10), equation (A12) yields:

$$R_{uy}(t_1,t_2) = R_{yu}(t_1,t_2) = \sigma_u^2 a_1 \rho_u \tag{A14}$$

When the input statistics are stationary this may be written as:

$$R_{uy}(\tau) = R_{yu}(\tau) = \sigma_u^2 a_1 \rho_u \tag{A15}$$

where

$$\tau = t_1 - t_2$$

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